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# Global smooth solutions for the quasilinear wave equation with boundary dissipation <sup>☆</sup>

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## Abstract

We consider the existence of global solutions of the quasilinear wave equation with a boundary dissipation structure of an input–output in high dimensions when initial data and boundary inputs are near a given equilibrium of the system. Our main tool is the geometrical analysis. The main interest is to study the effect of the boundary dissipation structure on solutions of the quasilinear system. We show that the existence of global solutions depends not only on this dissipation structure but also on a Riemannian metric, given by the coefficients and the equilibrium of the system. Some geometrical conditions on this Riemannian metric are presented to guarantee the existence of global solutions. In particular, we prove that the norm of the state of the system decays exponentially if the input stops after a finite time, which implies the exponential stabilization of the system by boundary feedback.

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## 1. Introduction and the main results

Let  $n \geq 2$  be an integer. Let  $\Omega \subset \mathbb{R}^n$  be a bounded, open set with the smooth boundary  $\Gamma = \Gamma_0 \cup \Gamma_1$  and  $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ . Let

$$\mathbf{a}(x, y) = (a_1(x, y), \dots, a_n(x, y)) : \overline{\Omega} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

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be a smooth mapping with

$$\mathbf{a}(x, 0) = 0, \quad x \in \overline{\Omega}, \quad (1.1)$$

such that  $(a_{ij}(x, y))$  is symmetrical and

$$(a_{ij}(x, y)) > 0, \quad \forall (x, y) \in \overline{\Omega} \times \mathbb{R}^n, \quad (1.2)$$

where  $a_{ij} = a_{iy_j}$  are the partial derivatives of  $a_i$  with respect to the variable  $y$ . Let  $T > 0$  be given. We consider the following problem

$$\begin{cases} \ddot{u} = \operatorname{div} \mathbf{a}(x, \nabla u), & (t, x) \in (0, T) \times \Omega, \\ u|_{\Gamma_1} = 0, & t \in (0, T), \\ u(0, x) = u_0, \quad \dot{u}(0, x) = u_1, & x \in \Omega, \end{cases} \quad (1.3)$$

with an input  $I(t)$  and an output  $\mathcal{O}(t)$  on the portion  $\Gamma_0$  of the boundary  $\Gamma$

$$I(t) = \dot{u} + \lambda \langle \mathbf{a}(x, \nabla u), \nu \rangle, \quad (t, x) \in (0, T) \times \Gamma_0, \quad (1.4)$$

$$\mathcal{O}(t) = \dot{u} - \lambda \langle \mathbf{a}(x, \nabla u), \nu \rangle, \quad (t, x) \in (0, T) \times \Gamma_0, \quad (1.5)$$

where  $\langle \cdot, \cdot \rangle$  is the dot product of the Euclidean space  $\mathbb{R}^n$ ,  $\nu$  is the unit normal of  $\Gamma$ , and  $\lambda > 0$  is a constant.

If  $\sigma(x, y)$  is a smooth function on  $\overline{\Omega} \times \mathbb{R}^n$  such that  $\mathbf{a}(x, y) = (\sigma_{y_1}(x, y), \dots, \sigma_{y_n}(x, y))$ , we define the total energy of the quasilinear system at time  $t$  by

$$E(t) = 4\lambda \int_{\Omega} [\dot{u}^2/2 + \sigma(x, \nabla u)] dx \quad (1.6)$$

and obtain by a simple computation

$$\frac{dE(t)}{dt} = \int_{\Gamma_0} |I(t)|^2 d\Gamma - \int_{\Gamma_0} |\mathcal{O}(t)|^2 d\Gamma, \quad (1.7)$$

where  $u$  is a solution of the problem (1.3)–(1.4). This formula expresses that the rate of change of the energy is equal to the power supplied to the system by the input minus the power taken out by the output on  $\Gamma_0$ . The balance equation (1.7) means that the boundary structure (1.4) is dissipative if  $I = 0$ .

We say  $w$  is an equilibrium solution to the system (1.3) if

$$\operatorname{div} \mathbf{a}(x, \nabla w) = 0, \quad x \in \Omega, \quad w = 0, \quad x \in \Gamma_1. \quad (1.8)$$

It is well known that smooth solutions of quasilinear hyperbolic systems usually develop singularities after some time [17] and [12]. However, since the structure of the boundary dissipation (1.4) makes the energy dissipative, we expect that the introduction of the boundary structure (1.4) assures the existence of a global smooth solution when initial data and boundary inputs are near a given equilibrium. In fact, if the vector field  $\mathbf{a}$  does not depend on the variable  $x$ , i.e.,

$a_i(x, y) = a_i(y)$ , and if the given equilibrium is zero, the existence of global solutions under a boundary structure as in (1.4) where the input is zero has been given by [22]. In addition, a string with boundary dissipation was studied by [2,9,11,28], and many authors. Furthermore, for some specially quasilinearity, the problems were also studied by [1,3,8,13,18,19], and many authors. The aim of this paper is to seek general geometrical conditions for a quasilinear part as in (1.3) and for any equilibrium  $w$ , given by (1.8), to assure the system (1.3)–(1.4) to have global solutions when initial data and inputs are near  $w$ . The results in Theorems 1.1, 1.2, below show that such geometrical conditions are actually characteristics of a Riemannian metric  $g$ , given by (1.12) below, and the case of [22] has a classical metric where the metric (1.12) below is a constant matrix.

The geometrical conditions introduced here (**H1** below) are a substantial improvement over [22]. To explain the difference roughly, when  $a_i(x, y) = a_i(y)$  and the equilibrium is zero, the metric (1.12) below is a classical metric with zero sectional curvature and the classical calculus is enough. However, if  $a_i(x, y)$  depend on variable  $x$  or an equilibrium is nonzero, the metric (1.12) is nontrivial which forces an introduction of the geometric method (geometric analysis). The geometric method was introduced by [25] for the controllability of wave equations with variable coefficients and was extended by [4,5,15,16,20,26], and many authors. For a recent survey on the geometric method, see [10].

Let

$$m \geq [n/2] + 3$$

be a given positive integer. We introduce a Banach space for inputs. Let  $\mathcal{I}^m((0, \infty), \Gamma_0)$  consist of all the functions  $I(t) = I(t, x)$  on  $(0, \infty) \times \Gamma_0$  such that

$$\begin{aligned} I^{(k)}(t) &\in L^2((0, \infty), H^{m-k-3/2}(\Gamma_0)) \cap C[0, \infty; H^{m-k-3/2}(\Gamma_0)], \quad 0 \leq k \leq m-2, \\ I^{(m-1)}(t) &\in L^2((0, \infty), L^2(\Gamma_0)), \end{aligned}$$

with a norm

$$\|I\|_{\mathcal{I}^m((0, \infty), \Gamma_0)}^2 = \max_{0 \leq t < \infty} \mathcal{E}_{I, \Gamma_0}(t) + \int_0^\infty \mathcal{E}_{I, \Gamma_0}(\tau) d\tau + \int_0^\infty \|I^{(m-1)}(\tau)\|_{L^2(\Gamma_0)}^2 d\tau \quad (1.9)$$

where  $\mathcal{E}_{I, \Gamma_0}(t) = \sum_{k=0}^{m-2} \|I^{(k)}(t)\|_{H^{m-k-3/2}(\Gamma_0)}^2$ .

Let  $w \in H^m(\Omega) \cap H_{\Gamma_1}^1(\Omega)$  be an equilibrium of the system (1.3) where

$$H_{\Gamma_1}^1(\Omega) = \{v \in H^1(\Omega), v|_{\Gamma_1} = 0\}.$$

Inspired by [6], we hope to find solutions  $u(t, x)$  in

$$\bigcap_{k=0}^m C^k((0, \infty), H^{m-k}(\Omega)),$$

if

$$(u_0, u_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$$

is near  $(w, 0)$  in  $H^m(\Omega) \times H^{m-1}(\Omega)$  and  $I$  is in  $\mathcal{I}^m((0, \infty), \Gamma_0)$  and if these functions satisfy the following compatibility conditions

$$u_k|_{\Gamma_1} = 0, \quad 0 \leq k \leq m-1, \quad I^{(k)}(0) = I_k, \quad 0 \leq k \leq m-2,$$

where for  $k \geq 2$ ,

$$u_k = u^{(k)}(0), \quad (1.10)$$

as computed formally (and recursively) in terms of  $u_0$  and  $u_1$ , using the equation in (1.3), and for  $0 \leq k \leq m-2$

$$I_k = [\dot{u} + \lambda \langle \mathbf{a}(x, \nabla u), v \rangle]^{(k)}(0) \quad \text{on } \Gamma_0. \quad (1.11)$$

We define

$$g = A^{-1}(x, \nabla w), \quad (1.12)$$

where  $A(x, y) = (a_{ij}(x, y))$  for  $(x, y) \in \overline{\Omega} \times \mathbb{R}^n$ , as a Riemannian metric on  $\overline{\Omega}$  and consider the couple  $(\overline{\Omega}, g)$  as a Riemannian manifold with a boundary  $\Gamma$ . Here the metric  $g$  depends not only on the functions  $a_{ij}(\cdot, \cdot)$  and but also on the equilibrium  $w$ . We denote by  $\langle \cdot, \cdot \rangle_g$  the inner product induced by  $g$ .

We make the following geometrical assumptions:

**H1** There is a  $C^1$  vector field  $H$  on  $\overline{\Omega}$  such that

$$D_g H(X, X) \geq \rho_0 |X|_g^2, \quad X \in \mathbb{R}_x^n, \quad x \in \overline{\Omega}, \quad (1.13)$$

where  $D_g$  is the Levi-Civita connection of the metric  $g$ ,  $D_g H$  is the covariant differential of  $H$  of the metric  $g$ , and  $\rho_0 > 0$  is a constant.

**H2** The boundary portions  $\Gamma_1$  and  $\Gamma_0$  satisfy

$$\langle H, v \rangle \leq 0, \quad x \in \Gamma_1, \quad (1.14)$$

$$\langle H, v \rangle \geq 0, \quad x \in \Gamma_0. \quad (1.15)$$

Roughly speaking, the assumptions (1.13) and (1.14) will force a ray started from the domain to hit the portion  $\Gamma_0$  after some time, see [23], and, therefore, the dissipative structure (1.4) can eliminate the resonance which is introduced by initial data near the equilibrium  $w$ , to assure the existence of a global solution. The condition (1.13) may be not true and a counterexample is given in [25]. If  $\Gamma_1 = \emptyset$ , the condition (1.15) is star-shaped. This condition can be removed if an estimate of boundary trace can be established for the quasilinear wave equation as that for the linear wave equation in [14].

Let  $h$  be a strictly convex function of the metric  $g$  on  $\overline{\Omega}$ . Then  $H = \nabla_g h$  satisfies the condition (1.13) for some  $\rho_0 > 0$  where  $\nabla_g$  is the gradient of the metric  $g$ . However, the existence of such a strictly convex function  $h$  depends on the sectional curvature of the Riemannian manifold  $(\overline{\Omega}, g)$  closely. One of candidates for strictly convex functions is the distance function of the metric  $g$ . Let  $x_0 \in \Omega$  be given. Let  $\rho$  denote the distance function of the metric  $g$  from  $x_0 \in \Omega$  to  $x \in \Omega$ .

If  $a_i(x, y) = a_i(y)$  and if the equilibrium is zero, then  $g$  is a constant matrix and the Riemannian manifold  $(\overline{\Omega}, g)$  has zero sectional curvature. In this case

$$D_g^2(\rho^2)(X, X) = 2|X|_g^2, \quad X \in \mathbb{R}_x^n, \quad x \in \overline{\Omega}.$$

Thus  $H = 2\rho \nabla_g \rho$  satisfies the assumption **H1** with  $\rho_0 = 2$ . For a general case where  $g$  is not a constant matrix, there are a number of methods and examples in [25] to find out if  $H = 2\rho \nabla_g \rho$  satisfies the assumption **H1**.

We assume that solutions of short time exist for the system (1.3)–(1.4) with appropriate initial data and boundary input. In fact, there are standard approaches to obtain solutions of short time, for example, see [6] or [21].

We shall establish the following

**Theorem 1.1.** *Let an equilibrium  $w \in H_{\Gamma_1}^m(\Omega)$  be such that the assumptions **H1** and **H2** hold. Let  $(u_0, u_1)$  and  $\mathbf{I}$  satisfy the compatibility conditions. If  $\Gamma_1 = \emptyset$ , we further assume that*

$$\int_{\Gamma} u_0 d\Gamma = \int_{\Omega} w dx. \quad (1.16)$$

*Then for any  $(u_0 - w, u_1)$  in  $(H^m(\Omega) \cap H_{\Gamma_1}^1(\Omega)) \times (H^{m-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega))$  small and any  $\mathbf{I} - \lambda \langle \mathbf{a}(x, \nabla w), v \rangle$  in  $\mathcal{I}^m((0, \infty), \Gamma_0)$  small, the system (1.3)–(1.4) has a global solution  $u$  in*

$$\bigcap_{k=0}^m C^k((0, \infty), H^{m-k}(\Omega)).$$

We need the assumption (1.16) for a uniqueness result in Section 3 of this paper to get rid of some lower order terms when  $\Gamma_1 = \emptyset$ , see Lemma 3.2. Instead of (1.16) there are other options.

If the input  $\mathbf{I}(t) = \lambda \langle \mathbf{a}(x, \nabla w), v \rangle$  after a finite time  $T_0 > 0$ , the energy will exponentially decay, that is the stabilization by feedback from boundary.

**Theorem 1.2.** *Let all the assumptions of Theorem 1.1 hold. If there is  $T_0 > 0$  such that  $\mathbf{I}(t) = \lambda \langle \mathbf{a}(x, \nabla w), v \rangle$  for  $t \geq T_0$ , then there are  $c_1 > 0$ ,  $c_2 > 0$ , and  $\hat{T} \geq T_0$  such that*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 t}, \quad t \geq \hat{T}, \quad (1.17)$$

where  $\mathcal{E}(t) = \|u - w\|_{H^m(\Omega)}^2 + \sum_{k=1}^m \|u^{(k)}\|_{H^{m-k}(\Omega)}^2$ .

We mention that in the linear case the structure of dissipation, as above, has been studied thoroughly, for example, see [24].

Finally, let us see some examples with nonzero equilibria. Let

$$\mathbf{a}(y) = (a_1(y_1), a_2(y_2)) : \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

where  $a_i$  are smooth functions on  $\mathbb{R}$  for  $i = 1, 2$  such that

$$a_i'(s) > 0, \quad s \in \mathbb{R}.$$

Let

$$w(x) = x_1 x_2, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$\operatorname{div} \mathbf{a}(\nabla w) = 0, \quad x \in \mathbb{R}^2.$$

The metric is

$$g = A^{-1}(x, \nabla w) = \begin{pmatrix} 1/a_1'(x_2) & 0 \\ 0 & 1/a_2'(x_1) \end{pmatrix}. \quad (1.18)$$

We let

$$a_1(s) = a_2(s) = \arctan s, \quad s \in \mathbb{R}.$$

Then

$$g = \begin{pmatrix} 1+x_2^2 & 0 \\ 0 & 1+x_1^2 \end{pmatrix}.$$

By [25, Lemma 3.2], the Gauss curvature of  $(\mathbb{R}^2, g)$  is

$$\kappa(x) = -\frac{2+|x|^2}{(1+x_1^2)^2(1+x_2^2)^2} \leq 0, \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (1.19)$$

By [25, Corollary 1.2], the assumption **H1** in (1.13) holds true for any  $\Omega \subset \mathbb{R}^2$ .

If we let

$$a_1(s) = a_2(s) = \frac{1}{3}s^3 + s,$$

then

$$g = \begin{pmatrix} \frac{1}{1+x_2^2} & 0 \\ 0 & \frac{1}{1+x_1^2} \end{pmatrix},$$

and

$$\kappa(x) = \frac{1-2x_1^2}{(1+x_1^2)^6(1+x_2^2)} + \frac{1-2x_2^2}{(1+x_1^2)(1+x_2^2)^6}.$$

If

$$\overline{\Omega} \subset \mathbb{R}^2 / \{x \mid |x_1| > 1/2, |x_2| > 1/2\},$$

then **H1** is true.

## 2. Estimates of boundary trace I

We assume that solutions to the system (1.3)–(1.4) exist for some  $T > 0$  near a given equilibrium  $w \in H^m(\Omega)$  with  $w|_{\Gamma_1} = 0$  and write those as  $u = w + \phi$ . Noting that  $w$  does not depend on  $t$ , the system (1.3)–(1.4) is equivalent to

$$\begin{cases} \ddot{\phi} = \operatorname{div} \mathbf{b}(x, \nabla \phi), & (t, x) \in (0, T) \times \Omega, \\ \phi = 0, & (t, x) \in (0, T) \times \Gamma_1, \\ \mathbf{I}(t) = \dot{\phi} + \lambda \langle \mathbf{b}(x, \nabla \phi), \nu \rangle, & (t, x) \in (0, T) \times \Gamma_0, \\ \phi(0) = \phi_0, \quad \dot{\phi}(0) = \phi_1, \end{cases} \quad (2.1)$$

with the output

$$\mathcal{O}(t) = \dot{\phi} - \lambda \langle \mathbf{b}(x, \nabla \phi), \nu \rangle, \quad (t, x) \in (0, T) \times \Gamma_0, \quad (2.2)$$

where we have set

$$\mathbf{b}(x, y) = \mathbf{a}(x, \nabla w + y), \quad (x, y) \in \overline{\Omega} \times \mathbb{R}^n, \quad (2.3)$$

$$\phi_0 = u_0 - w, \quad \phi_1 = u_1, \quad x \in \Omega. \quad (2.4)$$

In this section we establish some estimates of boundary trace for solutions  $\phi$  of the system (2.1) on  $\mathcal{D}_0 = (0, T) \times \Gamma_0$ .

Let  $\phi$  solve the problem (2.1) for some  $T > 0$ . Let

$$B(x, y) = (b_{ij}(x, y)), \quad b_{ij}(x, y) = a_{ij}(x, \nabla w + y), \quad (x, y) \in \overline{\Omega} \times \mathbb{R}^n. \quad (2.5)$$

Then

$$\dot{\mathbf{b}}(x, \nabla \phi) = B_\phi(t) \nabla \dot{\phi}, \quad (2.6)$$

and for  $j \geq 2$

$$\mathbf{b}^{(j)}(x, \nabla \phi) = B_\phi(t) \nabla \phi^{(j)} + \sum_{k=1}^{j-1} B_\phi^{(k)}(t) \nabla \phi^{(j-k)}, \quad (2.7)$$

where

$$B_\phi(t) = B(x, \nabla \phi). \quad (2.8)$$

We define

$$\mathcal{B}_\phi(t)v = \operatorname{div} B_\phi(t) \nabla v, \quad v \in H^2(\Omega), \quad (2.9)$$

and

$$v_{\nu_B} = \langle B_\phi(t) \nabla v, \nu \rangle, \quad v \in H^2(\Omega), \quad x \in \Gamma. \quad (2.10)$$

Then

$$(\mathcal{B}_\phi(t)v, \varphi)_{L^2(\Omega)} = -(B_\phi(t)\nabla v, \nabla \varphi)_{L^2(\Omega)} + \int_{\Gamma} \varphi v_{v_B} d\Gamma, \quad v, \varphi \in H^2(\Omega). \quad (2.11)$$

Moreover, we introduce an operator

$$\mathcal{N}_\phi(t)v = \operatorname{div} N_\phi(t)\nabla v, \quad v \in H^2(\Omega), \quad (2.12)$$

where

$$N_\phi(t) = \int_0^1 B(x, s\nabla \phi) ds. \quad (2.13)$$

Then the system (2.1) can be rewritten as

$$\begin{cases} \ddot{\phi}(t) = \mathcal{N}_\phi(t)\phi(t), & (t, x) \in (0, T) \times \Omega, \\ \phi = 0, & (t, x) \in (0, T) \times \Gamma_1, \\ \phi(0) = \phi_0, \quad \dot{\phi}(0) = \phi_1, & x \in \Omega, \end{cases} \quad (2.14)$$

with the following input–output relation

$$\begin{cases} \dot{\phi} + \lambda\phi_{v_N} = \mathbf{I}(t) - \lambda\langle \mathbf{a}(x, \nabla w), v \rangle, & x \in \Gamma_0, \\ \dot{\phi} - \lambda\phi_{v_N} = \mathcal{O}(t) + \lambda\langle \mathbf{a}(x, \nabla w), v \rangle, & x \in \Gamma_0. \end{cases} \quad (2.15)$$

Next, we define

$$\mathcal{E}(t) = \sum_{k=0}^m \|\phi^{(k)}(t)\|_{H^{m-k}(\Omega)}^2, \quad (2.16)$$

$$\begin{aligned} \mathcal{Q}(t) &= 2\lambda[\|\dot{\phi}(t)\|_{L^2(\Omega)}^2 + (N_\phi(t)\nabla \phi, \nabla \phi)_{L^2(\Omega)}] \\ &\quad + 2\lambda \sum_{k=1}^{m-1} [\|\phi^{(k+1)}(t)\|_{L^2(\Omega)}^2 + (B_\phi(t)\nabla \phi^{(k)}(t), \nabla \phi^{(k)}(t))_{L^2(\Omega)}], \end{aligned} \quad (2.17)$$

$$P(t) = 4\lambda[\|\dot{\phi}(t)\|_{L^2(\Omega)}^2 + (N_\phi(t)\nabla \phi, \nabla \phi)_{L^2(\Omega)} + \|\phi(t)\|_{L^2(\Omega)}^2], \quad (2.18)$$

$$\mathcal{Q}_{\mathbf{I}, \Gamma_0}(t) = \|\mathbf{I} - \lambda\langle \mathbf{a}(x, \nabla w), v \rangle\|_{L^2(\Gamma_0)}^2 + \sum_{k=1}^{m-1} \|\mathbf{I}^{(k)}(t)\|_{L^2(\Gamma_0)}^2, \quad (2.19)$$

$$\mathcal{Q}_{\mathcal{O}, \Gamma_0}(t) = \|\mathcal{O}(t) + \lambda\langle \mathbf{a}(x, \nabla w), v \rangle\|_{L^2(\Gamma_0)}^2 + \sum_{k=1}^{m-1} \|\mathcal{O}^{(k)}(t)\|_{L^2(\Gamma_0)}^2, \quad (2.20)$$

$$\mathcal{E}_{\mathbf{I}, \Gamma_0}(t) = \|\mathbf{I} - \lambda\langle \mathbf{a}(x, \nabla w), v \rangle\|_{H^{m-3/2}(\Gamma_0)}^2 + \sum_{k=1}^{m-2} \|\mathbf{I}^{(k)}(t)\|_{H^{m-k-3/2}(\Gamma_0)}^2, \quad (2.21)$$



$$\mathcal{L}(t) = \sum_{k=3}^{2m} \mathcal{E}^{k/2}(t). \quad (2.22)$$

We have

**Theorem 2.1.** *Let  $\gamma > 0$  be given and  $\phi$  satisfy the problem (2.1) on the interval  $[0, T]$  for some  $T > 0$  such that*

$$\sup_{0 \leq t \leq T} \|\phi(t)\|_{H^m(\Omega)} \leq \gamma. \quad (2.23)$$

*Then there are  $c_{0,\gamma} > 0$  and  $c_\gamma > 0$ , which depend on  $\gamma$ , such that*

$$c_{0,\gamma} \mathcal{Q}(t) \leq \mathcal{E}(t) \leq c_\gamma \mathcal{Q}(t) + c_\gamma \mathcal{E}_{1,\Gamma_0}(t) + c_\gamma \|\phi(t)\|_{L^2(\Omega)}^2 + c_\gamma \mathcal{L}(t), \quad (2.24)$$

$$\dot{\mathcal{Q}}(t) \leq 4\mathcal{Q}_{1,\Gamma_0}(t) - \mathcal{Q}_{\mathcal{O},\Gamma_0}(t) + c_\gamma \mathcal{L}(t), \quad (2.25)$$

$$-\dot{\mathcal{Q}}(t) \leq 2\mathcal{Q}_{1,\Gamma_0}(t) + 2\mathcal{Q}_{\mathcal{O},\Gamma_0}(t) + c_\gamma \mathcal{L}(t), \quad (2.26)$$

$$P(t) \leq \left\{ P(0) + c_\gamma \int_0^T [\mathcal{E}_{1,\Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau \right\} e^t, \quad (2.27)$$

for  $0 \leq t \leq T$ .

We collect here a few basic properties of Sobolev spaces to be invoked in the sequel.

(i) Let  $s_1 > s_2 \geq 0$ . For any  $\varepsilon > 0$  there is  $c_\varepsilon > 0$  such that

$$\|v\|_{H^{s_2}(\Omega)}^2 \leq \varepsilon \|v\|_{H^{s_1}(\Omega)}^2 + c_\varepsilon \|v\|_{L^2(\Omega)}^2 \quad \forall v \in H^{s_1}(\Omega). \quad (2.28)$$

(ii) If  $s > n/2$ , then for each  $k = 0, \dots$ , we have  $H^{s+k}(\Omega) \subset C^k(\overline{\Omega})$  with continuous inclusion.

(iii) If  $r := \min\{s_1, s_2, s_1 + s_2 - [n/2] - 1\} \geq 0$ , then there is a constant  $c > 0$  such that

$$\|f_1 f_2\|_{H^r(\Omega)} \leq c \|f_1\|_{H^{s_1}(\Omega)} \|f_2\|_{H^{s_2}(\Omega)} \quad \forall f_1 \in H^{s_1}(\Omega), f_2 \in H^{s_2}(\Omega). \quad (2.29)$$

(iv) Let  $s_j \geq 0$ ,  $j = 1, \dots, k$ , and  $r := \min_{1 \leq i \leq k} \min_{j_1 \leq \dots \leq j_i} \{s_{j_1} + \dots + s_{j_i} - (i-1) \times ([n/2] + 1)\} \geq 0$ . Then there a constant  $c > 0$  such that

$$\|f_1 \cdots f_k\|_{H^r(\Omega)} \leq c \|f_1\|_{H^{s_1}(\Omega)} \cdots \|f_k\|_{H^{s_k}(\Omega)} \quad \forall f_j \in H^{s_j}(\Omega), 1 \leq j \leq k. \quad (2.30)$$

**Lemma 2.1.** *Let  $\gamma > 0$  be given and  $\phi \in H^1(\Omega)$  be such that*

$$\sup_{x \in \overline{\Omega}} |\nabla \phi| \leq \gamma.$$

Let  $f(\cdot, \cdot)$  be a smooth function on  $\overline{\Omega} \times \mathbb{R}^n$ . Let  $F(x) = f(x, \nabla\phi)$ . Then there is  $c_\gamma > 0$ , depending on  $\gamma$ , such that

$$\|F\|_{H^k(\Omega)} \leq c_\gamma \sum_{j=1}^k (1 + \|\phi\|_{H^m(\Omega)})^j, \quad (2.31)$$

for  $0 \leq k \leq m-1$ . Moreover, if  $f(x, 0) = 0$  for  $x \in \Omega$  and  $\|\phi\|_{H^m(\Omega)} \leq \gamma$ , then there is  $c_\gamma > 0$  such that

$$\|F\|_{H^k(\Omega)} \leq c_\gamma \|\phi\|_{H^{k+1}(\Omega)}, \quad 0 \leq k \leq m-1. \quad (2.32)$$

**Proof.** The inequality (2.31) is given by [27, Lemma 2.1]. We here prove the inequality (2.32). Let  $f(x, 0) = 0$ . Then

$$F(x) = \sum_{j=1}^n f_j(x, \nabla\phi) \phi_{x_j}, \quad f_j(x, \nabla\phi) = \int_0^1 f_{y_j}(x, \sigma \nabla\phi) d\sigma.$$

Applying the inequalities (2.29) and (2.31) to the above formula yields

$$\|F\|_{H^k(\Omega)} \leq \sum_{j=1}^n \|f_j(\cdot, \nabla\phi)\|_{H^{m-1}(\Omega)} \|\phi_{x_j}\|_{H^k(\Omega)} \leq c_\gamma \|\phi\|_{H^{k+1}(\Omega)}. \quad \square$$

**Lemma 2.2.** Let  $\gamma > 0$  be given and  $\phi$  satisfy the problem (2.1) on the interval  $[0, T]$  for some  $T > 0$  such that the condition (2.23) is true. For  $1 \leq j \leq m-2$ , let

$$r_j(t) = \sum_{k=1}^j \operatorname{div} B_\phi^{(k)}(t) \nabla \phi^{(j-k)}, \quad r_{j, \Gamma_0}(t) = \sum_{k=1}^j \langle B_\phi^{(k)}(t) \nabla \phi^{(j-k)}(t), \nu \rangle, \quad (2.33)$$

where the matrix  $B_\phi(t)$  is given by (2.8). Then

$$\|r_j(t)\|_{H^{m-2-j}(\Omega)}^2, \quad \|r_{j, \Gamma_0}(t)\|_{H^{m-3/2-j}(\Gamma_0)}^2 \leq c_\gamma \sum_{k=2}^{m-1} \mathcal{E}^k(t), \quad 1 \leq j \leq m-2. \quad (2.34)$$

**Proof.** We have

$$b_{pq}^{(k)}(x, \nabla\phi) = \sum_{i=1}^k \sum_{r_1+\dots+r_i=k} D_y^i b_{pq}(\nabla\phi^{(r_1)}(t), \dots, \nabla\phi^{(r_i)}(t)), \quad 1 \leq k \leq j, \quad (2.35)$$

where  $D_y^i b_{pq}$  denotes the covariant differential of  $i$  order of the function  $b_{pq}(x, y)$ , given in (2.5), with respect to the variable  $y$  in the dot metric of  $\mathbb{R}^n$ . Thus,  $r_j(t)$  is a sum of some functions in the form

$$(f(x, \nabla\phi) \phi_{x_{j_1}}^{(r_1)} \dots \phi_{x_{j_i}}^{(r_i)} \phi_{x_p}^{(j-k)})_{x_q} \quad (2.36)$$

with  $r_1 + \dots + r_i = k$  for  $1 \leq i \leq k$ . Using the estimates (2.30) and (2.31), we have

$$\begin{aligned} & \left\| (f(x, \nabla \phi) \phi_{x_{j_1}}^{(r_1)} \dots \phi_{x_{j_i}}^{(r_i)} \phi_{x_p}^{(j-k)})_{x_q} \right\|_{H^{m-2-j}(\Omega)}^2 \\ & \leq \left\| f(x, \nabla \phi) \phi_{x_{j_1}}^{(r_1)} \dots \phi_{x_{j_i}}^{(r_i)} \phi_{x_p}^{(j-k)} \right\|_{H^{m-1-j}(\Omega)}^2 \leq c_\gamma \mathcal{E}^{i+1}(t), \end{aligned} \quad (2.37)$$

for  $1 \leq i \leq k \leq j \leq m-2$ . The estimate of  $\|r_j(t)\|_{H^{m-2-j}(\Omega)}^2$  in (2.34) follows from (2.37).

After we extend the domain of  $v$  from  $\Gamma_0$  to the whole  $\overline{\Omega}$ , a similar argument yields the estimate of  $\|r_{j, \Gamma_0}(t)\|_{H^{m-3/2-j}(\Gamma_0)}^2$ .  $\square$

We introduce a metric on  $\overline{\Omega}$  by

$$g_\phi = B_\phi^{-1}(t), \quad x \in \overline{\Omega}, \quad (2.38)$$

where matrix  $B_\phi(t)$  is given in (2.8). Then

$$\Delta_{g_\phi} v = \mathcal{B}_\phi(t)v + \frac{1}{2G} \sum_{ij} b_{ij}(x, \nabla \phi) G_{x_i} v_{x_j}, \quad G = (\det B_\phi(t))^{-1}, \quad v \in H^2(\Omega), \quad (2.39)$$

where  $\Delta_{g_\phi}$  is the Laplacian of the matrix  $g_\phi$  and the operator  $\mathcal{B}_\phi(t)$  is defined by (2.9). It follows from (2.39) and Lemma 2.1 that

$$\|\Delta_{g_\phi} v\|_k - c_{0\gamma} \|v\|_{k+1} \leq \|\mathcal{B}_\phi(t)v\|_k \leq \|\Delta_{g_\phi} v\|_k + c_\gamma \|v\|_{k+1}, \quad 0 \leq k \leq m-1. \quad (2.40)$$

In addition,

$$\langle X, Y \rangle_{g_\phi} = \langle B_\phi^{-1}(t)X, Y \rangle, \quad (2.41)$$

where  $X, Y$  are vector fields on  $\overline{\Omega}$  and  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_{g_\phi}$  are the products of the dot metric and the metric  $g_\phi$ , respectively.

We extend the definition of the normal  $v$  from  $\Gamma$  to  $\overline{\Omega}$ , still denoted by  $v$ , such that

$$|v| = 1, \quad x \in \overline{\Omega},$$

and set

$$N = \frac{B_\phi(t)v}{|B_\phi^{1/2}(t)v|}, \quad x \in \overline{\Omega}. \quad (2.42)$$

Then  $N$  is an unit vector field on  $\overline{\Omega}$  in the metric  $g$  and

$$N(v) = \frac{v v_B}{|B_\phi^{1/2}(t)v|}, \quad x \in \Gamma.$$

**Lemma 2.3.** Let  $\gamma > 0$  be given and  $\phi$  satisfy the problem (2.1) on the interval  $[0, T]$  for some  $T > 0$  such that the condition (2.23) is true. Then there is  $c_\gamma > 0$ , which depends on  $\gamma$ , such that

$$\begin{aligned} \|v\|_{H^{k+1}(\Omega)}^2 &\leq c_\gamma (\|\mathcal{B}_\phi(t)v\|_{H^{k-1}(\Omega)}^2 + \|v_{v_B}\|_{H^{k-1/2}(\Gamma_0)}^2 + \|v\|_{H^k(\Omega)}^2), \\ v &\in H^{k+1}(\Omega) \cap H_{\Gamma_1}^1(\Omega), \quad 0 \leq k \leq m-1, \quad t \in [0, T]. \end{aligned} \quad (2.43)$$

**Proof.** By induction.

The formulas,

$$(B_\phi(t)\nabla v, \nabla v)_{L^2(\Omega)} = -(B_\phi(t)v, v)_{L^2(\Omega)} + (v, v_{v_B})_{L^2(\Gamma_0)},$$

imply that

$$\begin{aligned} c_\gamma \|\nabla v\|_{L^2(\Omega)}^2 &\leq \|\mathcal{B}_\phi(t)v\|_{H^{-1}(\Omega)} \|v\|_{H^1(\Omega)} + \|v\|_{H^{1/2}(\Gamma_0)} \|v_{v_B}\|_{H^{-1/2}(\Gamma_0)} \\ &\leq c_\varepsilon (\|\mathcal{B}_\phi(t)v\|_{H^{-1}(\Omega)}^2 + \|v_{v_B}\|_{H^{-1/2}(\Gamma_0)}^2) + \varepsilon \|v\|_{H^1(\Omega)}^2, \end{aligned}$$

for  $\varepsilon > 0$  small. Then the estimate (2.43) in the case  $k = 0$  follows.

We assume that the estimate (2.43) is true for some  $0 \leq k \leq m-2$  and shall prove that it is true with  $k$  replaced by  $k+1$ .

For  $v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ , let

$$\Phi(v) = \|\mathcal{B}_\phi(t)v\|_{H^k(\Omega)}^2 + \|v_{v_B}\|_{H^{k+1/2}(\Gamma_0)}^2 + \|v\|_{H^{k+1}(\Omega)}^2. \quad (2.44)$$

**Step I.** Let  $X$  be a vector field with a form of

$$X = \sum_{j=1}^n f_j(x, \nabla \phi) \frac{\partial}{\partial x_j}$$

such that

$$\langle X, N \rangle_{g_\phi} = 0. \quad (2.45)$$

Then the estimates in (2.30) and in Lemma 2.1 yield

$$\|[\mathcal{B}_\phi, X]v\|_{H^{k-1}(\Omega)} \leq c_\gamma \|v\|_{H^{k+1}(\Omega)}, \quad [\mathcal{B}_\phi, X] = \mathcal{B}_\phi X - X \mathcal{B}_\phi. \quad (2.46)$$

In addition, using some estimates as in (2.30) with  $H^{sj}(\Omega)$  replaced by  $H^{sj}(\Gamma_0)$  and with  $n$  replaced by  $n-1$ , we obtain

$$\|(X(v))_{v_B}\|_{H^{k-1/2}(\Gamma_0)} \leq c_\gamma \|v_{v_B}\|_{H^{k+1/2}(\Gamma_0)} + c_\gamma \|v\|_{H^{k+1}(\Omega)}, \quad (2.47)$$

where  $X(v)$  denotes the directional derivative of the function  $v$  along the vector field  $X$ , i.e.,  $X(v) = \sum_{j=1}^n f_j(x, \nabla \phi) v_{x_j}$ .

By the induction assumption and the inequalities (2.46)–(2.47), we have

$$\begin{aligned} \|X(v)\|_{H^{k+1}(\Omega)}^2 &\leq c_\gamma \|\mathcal{B}_\phi X(v)\|_{H^{k-1}(\Omega)}^2 + c_\gamma \|(X(v))_{v_B}\|_{H^{k-1/2}(\Gamma_0)}^2 + c_\gamma \|X(v)\|_{H^k(\Omega)}^2 \\ &\leq c_\gamma \|X(\mathcal{B}_\phi v)\|_{H^{k-1}(\Omega)}^2 + c_r \|v_{v_B}\|_{H^{k+1/2}(\Gamma_0)}^2 + c_\gamma \|v\|_{H^{k+1}(\Omega)}^2 \\ &\leq c_\gamma \Phi(v), \quad \forall v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega), \end{aligned} \quad (2.48)$$

since the relation (2.45) implies that  $X(w)|_{\Gamma_1} = 0$  when  $w|_{\Gamma_1} = 0$ .

**Step II.** Let us estimate  $\|N(v)\|_{H^{k+1}(\Omega)}$  for  $v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$  where  $N$  is given by (2.42). Since  $\|N(v)\|_{H^{k+1}(\Omega)}^2 = \|N(v)\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|(N(v))_{x_j}\|_{H^k(\Omega)}^2$ , we shall estimate  $\|(N(v))_{x_j}\|_{H^k(\Omega)}$  for  $1 \leq j \leq n$ .

We decompose the vector field  $\frac{\partial}{\partial x_j}$  into

$$\frac{\partial}{\partial x_j} = X_j + f_j N, \quad f_j = \left\langle \frac{\partial}{\partial x_j}, N \right\rangle_{g_\phi}, \quad \langle X_j, N \rangle_{g_\phi} = 0. \quad (2.49)$$

Then

$$(N(v))_{x_j} = NX_j(v) + f_j D_{g_\phi}^2 v(N, N) + \text{first order terms}, \quad (2.50)$$

where  $D_{g_\phi}^2(\cdot, \cdot)$  is the Hessian of the function  $v$  of the metric  $g_\phi$ .

By Step I, we obtain

$$\|NX_j(v)\|_{H^k(\Omega)}^2 \leq c_\gamma \|X_j(v)\|_{H^{k+1}(\Omega)}^2 \leq c_\gamma \Phi(v), \quad \forall v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega). \quad (2.51)$$

Next, we estimate  $D_{g_\phi}^2 v(N, N)$ . Let  $E_1, \dots, E_n$  be an orthonormal frame on  $\overline{\Omega}$  in the dot metric such that  $E_n = B^{-1/2}(x, \nabla \phi)N$ . Then  $Y_1, \dots, Y_n$  is an orthonormal frame on  $\overline{\Omega}$  in the metric  $g_\phi$  where  $Y_i = B^{1/2}(x, \nabla \phi)E_i$  for  $1 \leq i \leq n$ . We have

$$D_{g_\phi}^2 v(N, N) = \Delta_{g_\phi} v - \sum_{i=1}^{n-1} Y_i Y_i(v) + \text{first order terms}. \quad (2.52)$$

Then, by (2.40) and Step I again,

$$\begin{aligned} \|D_{g_\phi}^2 v(N, N)\|_{H^k(\Omega)}^2 &\leq c_\gamma \|\mathcal{B}_\phi(v)\|_{H^k(\Omega)}^2 + c_\gamma \sum_{i=1}^{n-1} \|Y_i(v)\|_{H^{k+1}(\Omega)}^2 + c_\gamma \|v\|_{H^{k+1}(\Omega)}^2 \\ &\leq c_\gamma \Phi(v), \quad \forall v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega). \end{aligned} \quad (2.53)$$

Combining (2.50), (2.51) and (2.53) together gives

$$\|N(v)\|_{H^{k+1}(\Omega)}^2 \leq c_\gamma \Phi(v), \quad \forall v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega). \quad (2.54)$$

**Step III.** From the inequalities (2.48) and (2.54), we have

$$\begin{aligned}\|v\|_{H^{k+2}(\Omega)}^2 &= \|v\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|v_{x_j}\|_{H^{k+1}(\Omega)}^2 \\ &\leq c_\gamma \sum_{j=1}^n \|X_j(v)\|_{H^{k+1}(\Omega)}^2 + c_\gamma \|N(v)\|_{H^{k+1}(\Omega)}^2 + c_\gamma \|v\|_{H^{k+1}(\Omega)}^2 \\ &\leq c_\gamma \Phi(v), \quad \forall v \in H^{k+2}(\Omega) \cap H_{\Gamma_1}^1(\Omega),\end{aligned}\quad (2.55)$$

that is, the inequality (2.43) holds when  $k$  is replaced with  $k+1$ .

Lemma 2.3 follows by induction.  $\square$

**Proof of Theorem 2.1.** The left-hand side of the inequality (2.24) is clearly true. We prove the right-hand side of it. We have

$$\begin{aligned}\|\phi^{(m)}(t)\|_{L^2(\Omega)}^2 + \|\phi^{(m-1)}(t)\|_{H^1(\Omega)}^2 \\ \leq \|\phi^{(m)}(t)\|_{L^2(\Omega)}^2 + c_\gamma (B_\phi(t) \nabla \phi^{(m-1)}, \nabla \phi^{(m-1)})_{L^2(\Omega)} + \|\phi^{(m-1)}(t)\|_{L^2(\Omega)}^2 \\ \leq c_\gamma \mathcal{Q}(t).\end{aligned}\quad (2.56)$$

Proceeding by induction, we assume that for  $3 \leq l \leq m$ , when  $j=l$  and  $j=l-1$ ,

$$\|\phi^{(j)}(t)\|_{H^{m-j}(\Omega)}^2 \leq c_\gamma \mathcal{Q}(t) + c_\gamma \mathcal{E}_{1,\Gamma_0}(t) + c_\gamma \|\phi(t)\|_{L^2(\Omega)}^2 + c_\gamma \mathcal{L}(t), \quad (2.57)$$

which, as shown above, is true for  $l=m$ . Formal differentiation of the equations in (2.1) and (2.2)  $l-2$  times with respect to  $t$  and use of the formulas (2.6)–(2.7) yield

$$\begin{cases} \phi^{(l)}(t) = \mathcal{B}_\phi(t) \phi^{(l-2)}(t) + r_{l-3}(t), & (t, x) \in (0, T) \times \Omega, \\ \phi^{(l-2)} = 0, & (t, x) \in (0, T) \times \Gamma_1, \\ \phi^{(l-1)}(t) + \lambda(\phi^{(l-2)})_{v_B}(t) + \lambda r_{l-3,\Gamma_0}(t) = \mathbf{I}^{(l-2)}(t), & (t, x) \in (0, T) \times \Gamma_0, \\ \phi^{(l-1)}(t) - \lambda(\phi^{(l-2)})_{v_B}(t) - \lambda r_{l-3,\Gamma_0}(t) = \mathcal{O}^{(l-2)}(t), & (t, x) \in (0, T) \times \Gamma_0, \end{cases} \quad (2.58)$$

where  $r_{l-3}(t)$  and  $r_{l-3,\Gamma_0}(t)$  are given in (2.33) if  $l \geq 4$ , and they are zero if  $l=3$ . Using Lemmas 2.3, 2.2, the formulas (2.57), (2.28), and the estimate (2.43), we obtain

$$\begin{aligned}\|\phi^{(l-2)}(t)\|_{H^{m-l+2}(\Omega)}^2 \\ \leq c_\gamma \|\mathcal{B}_\phi(t) \phi^{(l-2)}\|_{H^{m-l}(\Omega)}^2 + c_\gamma \|(\phi^{(l-2)})_{v_B}(t)\|_{H^{m-l+1/2}(\Gamma_0)}^2 + c_\gamma \|\phi^{(l-2)}(t)\|_{H^{m-l+1}(\Omega)}^2 \\ \leq c_\gamma \|\phi^{(l)}(t)\|_{H^{m-l}(\Omega)}^2 + c_\gamma \|\mathbf{I}^{(l-2)}(t)\|_{H^{m-l+1/2}(\Gamma_0)}^2 + c_\gamma \|\phi^{(l-1)}(t)\|_{H^{m-l+1}(\Omega)}^2 + c_\gamma \mathcal{L}(t) \\ + \varepsilon \|\phi^{(l-2)}(t)\|_{H^{m-l+2}(\Omega)}^2 + c_{\varepsilon,\gamma} \|\phi^{(l-2)}(t)\|_{L^2(\Omega)}^2,\end{aligned}$$

which implies that the inequality (2.57) is also true for  $j = l - 2$ . Then the inequalities (2.57) are true for  $1 \leq j \leq m$  by induction. On the other hand, using an ellipticity estimate for the operator  $\mathcal{N}_\phi(t)$  in (2.12) as that for  $\mathcal{B}_\phi$  in Lemma 2.3, we obtain, by the formulas (2.14),

$$\begin{aligned} \|\phi(t)\|_{H^m(\Omega)}^2 &\leq c_\gamma \|\mathcal{N}_\phi(t)\phi\|_{H^{m-2}(\Omega)}^2 + c_\gamma \|\phi_{v_N}\|_{H^{m-3/2}(\Gamma_0)}^2 + c_\gamma \|\phi(t)\|_{H^{m-1}(\Omega)}^2 \\ &\leq c_\gamma \|\phi^{(2)}(t)\|_{H^{m-2}(\Omega)}^2 + c_\gamma \|\dot{\phi}(t)\|_{H^{m-1}(\Omega)}^2 + c_{\gamma,\varepsilon} \|\phi(t)\|_{L^2(\Omega)}^2 \\ &\quad + c_\gamma \|\mathbf{I}(t) - \lambda \langle \mathbf{a}(x, \nabla w), v \rangle\|_{H^{m-3/2}(\Gamma_0)}^2 + \varepsilon \|\phi(t)\|_{H^m(\Omega)}^2. \end{aligned} \quad (2.59)$$

Combining (2.57) for  $1 \leq j \leq m$  and (2.59) yields the right-hand side of the inequality (2.24).

Let

$$\vartheta_0(t) = \|\dot{\phi}(t)\|_{L^2(\Omega)}^2 + (N_\phi(t) \nabla \phi, \nabla \phi)_{L^2(\Omega)}, \quad (2.60)$$

and, for  $1 \leq j \leq m - 1$ , let

$$\vartheta_j(t) = \|\phi^{(j+1)}(t)\|_{L^2(\Omega)}^2 + (B_\phi(t) \nabla \phi^{(j)}(t), \nabla \phi^{(j)}(t))_{L^2(\Omega)}. \quad (2.61)$$

For  $1 \leq j \leq m - 1$  using the formulas (2.11) and (2.58) for  $l = j + 2$ , we obtain

$$\begin{aligned} \dot{\vartheta}_j(t) &= 2(r_{j-1}(t), \phi^{(j+1)}(t))_{L^2(\Omega)} + 2(\phi^{(j+1)}(t), (\phi^{(j)})_{v_B}(t))_{L^2(\Gamma_0)} \\ &\quad + (\dot{B}_\phi(t) \nabla \phi^{(j)}(t), \nabla \phi^{(j)}(t))_{L^2(\Omega)} \\ &= (2\lambda)^{-1} (\|\mathbf{I}^{(j)}(t) - \lambda r_{j-1, \Gamma_0}(t)\|_{L^2(\Gamma_0)}^2 - \|\mathcal{O}^{(j)}(t) + \lambda r_{j-1, \Gamma_0}(t)\|_{L^2(\Gamma_0)}^2) \\ &\quad + 2(r_{j-1}(t), \phi^{(j+1)}(t))_{L^2(\Omega)} + (\dot{B}_\phi(t) \nabla \phi^{(j)}(t), \nabla \phi^{(j)}(t))_{L^2(\Omega)}. \end{aligned} \quad (2.62)$$

In the case  $j = 0$ , we use the formulas in (2.14) and (2.15) and obtain

$$\dot{\vartheta}_0(t) = (2\lambda)^{-1} (\|\mathbf{I} - \lambda \langle \mathbf{a}(x, \nabla w), v \rangle\|_{L^2(\Gamma_0)}^2 - \|\mathcal{O} + \lambda \langle \mathbf{a}(x, \nabla w), v \rangle\|_{L^2(\Gamma_0)}^2). \quad (2.63)$$

The inequalities (2.25) and (2.26) follow from (2.62) and (2.63) via Lemma 2.2 since  $\mathcal{Q}(t) = 2\lambda \sum_{j=0}^{m-1} \vartheta_j(t)$ .

Similarly, we have

$$\dot{P}(t) \leq \|\mathbf{I}(t) - \lambda \langle \mathbf{a}(x, \nabla w), v \rangle\|_{L^2(\Gamma_0)}^2 + c_\gamma \mathcal{E}^{3/2}(t) + P(t), \quad 0 \leq t \leq T,$$

which yields

$$P(t) \leq P(0) + c_\gamma \int_0^T [\mathcal{E}_{\Gamma_0}(\tau) + \mathcal{E}^{3/2}(\tau)] d\tau + \int_0^t P(\tau) d\tau \quad (2.64)$$

for  $0 \leq t \leq T$ . The inequality (2.27) follows from (2.64) by Gronwall's inequality.  $\square$

### 3. Estimates of boundary trace II

In this section, we shall establish the following

**Theorem 3.1.** *Let all the assumptions in Theorem 1.1 hold. Let  $\gamma > 0$  be given and  $\phi$  satisfy the problem (2.1) on the interval  $[0, T]$  for some  $T > 0$  such that (2.23) is true. Then there are  $c_\gamma > 0$  and  $T_\gamma > 3 \sup_{x \in \overline{\Omega}} |H|_g / \rho_0$ , where  $\rho_0 > 0$  is given by (1.13), such that, if  $0 \leq s \leq t \leq T$  are such that  $t - s \geq T_\gamma$ , then*

$$\int_s^t \mathcal{Q}(\tau) d\tau \leq c_\gamma \int_s^t [\mathcal{Q}_{\Gamma_0}(\tau) + \mathcal{Q}_{\partial, \Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau, \quad (3.1)$$

where  $\mathcal{Q}(t)$ ,  $\mathcal{Q}_{\Gamma_0}(t)$ ,  $\mathcal{Q}_{\partial, \Gamma_0}(t)$ , and  $\mathcal{L}(t)$  are given in (2.17), (2.19), and (2.21) of Section 2, respectively.

Let  $\gamma > 0$  be given and  $\phi$  satisfy the problem (2.1) on the interval  $[0, T]$  for some  $T > 0$  such that (2.23) is true. For  $t \in [0, T]$ , let  $g_\phi$  be the metric on  $\overline{\Omega}$  given by

$$g_\phi = B_\phi(t)^{-1}, \quad (3.2)$$

where the matrix  $B_\phi(t)$  is defined by (2.5) and (2.8) of Section 2 and consider the couple  $(\overline{\Omega}, g_\phi)$  as a Riemannian manifold for the fixed  $t$ . Let  $X, Y$  be vector fields on  $\overline{\Omega}$  and let  $f$  be a function. Then

$$\langle X, Y \rangle_{g_\phi} = \langle B_\phi^{-1}(t)X, Y \rangle, \quad \nabla_{g_\phi} f = B_\phi(t) \nabla f, \quad (3.3)$$

where  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_{g_\phi}$  are the products of the dot metric and the metric  $g_\phi$  and  $\nabla$  and  $\nabla_{g_\phi}$  are the gradients of the dot metric and the metric  $g_\phi$ , respectively. By (2.5), (2.8) and (3.2),

$$g_0 = A^{-1}(x, \nabla w) = g. \quad (3.4)$$

In addition, it is easy to check from the formulas in (3.3) that there are  $c_{0, \gamma} > 0$  and  $c_\gamma > 0$  such that

$$c_{0, \gamma} |\nabla_g f|_g^2 \leq |\nabla_{g_\phi} f|_{g_\phi}^2 = \langle B_\phi(t) \nabla f, \nabla f \rangle \leq c_\gamma |\nabla_g f|_g^2, \quad t \in [0, T], \quad f \in C^\infty(\Omega), \quad (3.5)$$

under the condition (2.23).

Let

$$(b_{ij}(x, y))^{-1} = (b^{ij}(x, y)), \quad (x, y) \in \overline{\Omega} \times \mathbb{R}^n, \quad (3.6)$$

where  $b_{ij}(x, y)$  are given in (2.5). Then

$$A^{-1}(x, \nabla w) = (b^{ij}(x, 0)). \quad (3.7)$$



Let  $D_{g_\phi}$  and  $D_g$  be the Levi-Civita connections of the Riemannian metrics  $g_\phi$  and  $g$ , respectively. Let  $H$  be a vector field on  $\overline{\Omega}$ . We denote by  $D_{g_\phi} H$  and by  $D_g H$  the covariant differentials of the metric  $g_\phi$  and  $g$ , respectively. They are two order tensor fields on  $\overline{\Omega}$ . We define

$$\eta = D_{g_\phi} H - D_g H. \quad (3.8)$$

**Lemma 3.1.** *Let  $H$  be a vector field on  $\overline{\Omega}$ . Suppose that the tensor field of order two  $\eta = \eta(\cdot, \cdot)$  is given by the formula (3.8). Let  $\gamma > 0$  be given and  $\phi$  be such that  $\sup_{x \in \overline{\Omega}} |\nabla \phi| \leq \gamma$ . Then there is  $c_\gamma > 0$  such that*

$$|\eta(X, Y)| \leq c_\gamma (|D\phi| + |D^2\phi|) |X| |Y|, \quad \forall X, Y \in \mathbb{R}_x^n, \quad x \in \overline{\Omega}, \quad (3.9)$$

where  $D$  is the covariant differential of the dot product of the Euclidean space  $\mathbb{R}^n$ .

**Proof.** Using the relations (2.41) and (3.4), we have

$$\begin{aligned} D_{g_\phi} H \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) &= \frac{\partial}{\partial x_j} \left\langle H, \frac{\partial}{\partial x_i} \right\rangle_{g_\phi} - \left\langle H, (D_{g_\phi})_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \right\rangle_{g_\phi} \\ &= \frac{\partial}{\partial x_j} \left\langle B_\phi^{-1}(t) H, \frac{\partial}{\partial x_i} \right\rangle - \sum_{k=1}^n \Gamma_{g_\phi i j}^k \left\langle B_\phi^{-1}(t) H, \frac{\partial}{\partial x_k} \right\rangle \\ &= D_g H \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) + \eta \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} \eta \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) &= \frac{\partial}{\partial x_j} \left[ \left( B_\phi^{-1}(t) - A^{-1}(x, \nabla w) \right) H, \frac{\partial}{\partial x_i} \right] - \sum_{k=1}^n \Gamma_{g_\phi i j}^k \left\langle B_\phi^{-1}(t) H, \frac{\partial}{\partial x_k} \right\rangle \\ &\quad + \sum_{k=1}^n \Gamma_{g i j}^k \left\langle A^{-1}(x, \nabla w) H, \frac{\partial}{\partial x_k} \right\rangle, \end{aligned} \quad (3.11)$$

$\Gamma_{g_\phi i j}^k$  and  $\Gamma_{g i j}^k$  are the coefficients of the connections  $D_{g_\phi}$  and  $D_g$ , respectively.

By definition, we have

$$\begin{aligned} \Gamma_{g_\phi i j}^k &= \frac{1}{2} \sum_{l=1}^n b_{lk}(x, \nabla \phi) \left[ (b^{lj}(x, \nabla \phi))_{\partial x_i} + (b^{li}(x, \nabla \phi))_{\partial x_j} - (b^{ij}(x, \nabla \phi))_{\partial x_l} \right] \\ &= f_{ijk}(x, \nabla \phi) + p_{ijk}(\phi), \end{aligned} \quad (3.12)$$

where

$$f_{ijk}(x, \nabla \phi) = \frac{1}{2} \sum_{l=1}^n b_{lk}(x, \nabla \phi) \left[ b_{x_j}^{il}(x, \nabla \phi) + b_{x_i}^{jl}(x, \nabla \phi) - b_{x_l}^{ij}(x, \nabla \phi) \right], \quad (3.13)$$

$$p_{ijk}(\phi) = \frac{1}{2} \sum_{lh=1}^n b_{lk}(x, \nabla\phi) [b_{yh}^{il}(x, \nabla\phi) \phi_{x_h x_j} + b_{yh}^{jl}(x, \nabla\phi) \phi_{x_h x_i} - b_{yh}^{ij}(x, \nabla\phi) \phi_{x_h x_l}]. \quad (3.14)$$

The formula (3.14) yields

$$|p_{ijk}(\phi)| \leq c_\gamma |D^2\phi|. \quad (3.15)$$

In addition, since  $b_{ij}(x, 0) = a_{ij}(x, \nabla w)$ , we have, by (3.13),

$$\Gamma_{gij}^k = f_{ijk}(x, 0). \quad (3.16)$$

Thus by (3.11)–(3.16),

$$\begin{aligned} & \sum_{k=1}^n \Gamma_{g\phi ij}^k \left\langle B_\phi^{-1}(t)H, \frac{\partial}{\partial x_k} \right\rangle \\ &= \sum_{k=1}^n \Gamma_{gij}^k \left\langle A^{-1}(x, \nabla w)H, \frac{\partial}{\partial x_k} \right\rangle + \sum_{k=1}^n [f_{ijk}(x, \nabla\phi) - f_{ijk}(x, 0)] \left\langle B_\phi^{-1}(t)H, \frac{\partial}{\partial x_k} \right\rangle \\ &+ \sum_{k=1}^n \Gamma_{gij}^k \left\langle [B_\phi^{-1}(t) - A^{-1}(x, \nabla w)]H, \frac{\partial}{\partial x_k} \right\rangle + \sum_{k=1}^n p_{ijk}(\phi) \left\langle B_\phi^{-1}(t)H, \frac{\partial}{\partial x_k} \right\rangle. \end{aligned} \quad (3.17)$$

Moreover, the relations

$$\begin{aligned} B_\phi^{-1}(t) - A^{-1}(x, \nabla w) &= \left( \sum_{k=1}^n \int_0^1 b_{y_k}^{ij}(x, \tau \nabla\phi) d\tau \phi_{x_k} \right), \\ f_{ijk}(x, \nabla\phi) - f_{ijk}(x, 0) &= \sum_{h=1}^n \int_0^1 f_{ijk y_h}(x, \tau \nabla\phi) d\tau \phi_{x_h}, \end{aligned}$$

imply that

$$\left| \left\langle [B_\phi^{-1}(t) - A^{-1}(x, \nabla w)]H, \frac{\partial}{\partial x_k} \right\rangle \right| \leq c_\gamma |D\phi|, \quad (3.18)$$

$$\left| \frac{\partial}{\partial x_j} \left\langle [B_\phi^{-1}(t) - A^{-1}(x, \nabla w)]H, \frac{\partial}{\partial x_i} \right\rangle \right| \leq c_\gamma (|D\phi| + |D^2\phi|), \quad (3.19)$$

$$|f_{ijk}(x, \nabla\phi) - f_{ijk}(x, 0)| \leq c_\gamma |D\phi|. \quad (3.20)$$

The inequality (3.9) follows from the formulas (3.11) and (3.17) and the estimates in (3.18)–(3.20), and (3.15).  $\square$

**Proof of Theorem 3.1.** Let  $\mathcal{H}$  be a vector field and let  $f$  be a function. We denote by  $\mathcal{H}(f)$  the directional derivative of the function  $f$  along the vector field  $\mathcal{H}$ . Then

$$\mathcal{H}(f) = \langle \mathcal{H}, \nabla f \rangle = \langle \mathcal{H}, \nabla_{g_\phi} f \rangle_{g_\phi}. \quad (3.21)$$

Let  $0 \leq s \leq t \leq T$ .

**Step I.** We assume that  $1 \leq j \leq m-1$ .

Letting  $l = j+2$  in (2.58), we have

$$\begin{cases} \ddot{\phi}^{(j)}(t) = \mathcal{B}_\phi(t)\phi^{(j)}(t) + r_{j-1}(t), & (t, x) \in (0, T) \times \Omega, \\ \phi^{(j)} = 0, & (t, x) \in (0, T) \times \Gamma_1, \\ \dot{\phi}^{(j)}(t) + \lambda\phi_{v_B}^{(j)}(t) + \lambda r_{j-1, \Gamma_0}(t) = \mathcal{I}^{(j)}(t), & (t, x) \in (0, T) \times \Gamma_0, \\ \dot{\phi}^{(j)}(t) - \lambda\phi_{v_B}^{(j)}(t) - \lambda r_{j-1, \Gamma_0}(t) = \mathcal{O}^{(j)}(t), & (t, x) \in (0, T) \times \Gamma_0, \end{cases} \quad (3.22)$$

where  $r_{j-1}(t)$  and  $r_{j-1, \Gamma_0}(t)$  are given in (2.33) if  $j \geq 2$ , and they are zero if  $j = 1$ .

We multiply the equations in (3.22) by  $\mathcal{H}(\phi^{(j)})$ , integrate by parts over  $\Sigma_s^t = (s, t) \times \Omega$ , and obtain (see [25, Proposition 2.1])

$$\begin{aligned} & \int_{\wp_s^t} \left[ \mathcal{H}(\phi^{(j)})\phi_{v_B}^{(j)} + \frac{1}{2}(\dot{\phi}^{(j)2} - |\nabla_{g_\phi}\phi^{(j)}|_{g_\phi}^2) \langle \mathcal{H}, v \rangle \right] d\wp \\ &= (\dot{\phi}^{(j)}, \mathcal{H}(\phi^{(j)}))_{L^2(\Omega)} \Big|_s^t - \int_{\Sigma_s^t} [\dot{\phi}^{(j)}\mathcal{H}(\phi^{(j)}) + r_{j-1}(t)\mathcal{H}(\phi^{(j)})] d\Sigma \\ &+ \int_{\Sigma_s^t} \left\{ D_{g_\phi}\mathcal{H}(\nabla_{g_\phi}\phi^{(j)}, \nabla_{g_\phi}\phi^{(j)}) + \frac{1}{2}[(\dot{\phi}^{(j)})^2 - |\nabla_{g_\phi}\phi^{(j)}|_{g_\phi}^2] \operatorname{div} \mathcal{H} \right\} d\Sigma. \end{aligned} \quad (3.23)$$

Let  $f \in C^2(\Omega)$ . We multiply the equations in (3.22) by  $f\phi^{(j)}$ , integrate by parts over  $\Sigma_s^t = (s, t) \times \Omega$ , and obtain

$$\begin{aligned} \int_{\Sigma_s^t} f[(\dot{\phi}^{(j)})^2 - |\nabla_{g_\phi}\phi^{(j)}|_{g_\phi}^2] d\Sigma &= (\dot{\phi}^{(j)}, f\phi^{(j)})_{L^2(\Omega)} \Big|_s^t + \int_{\wp_s^t} \left[ \frac{1}{2}(\phi^{(j)})^2 f_{v_B} - f\phi^{(j)}\phi_{v_B}^{(j)} \right] d\wp \\ &- \int_{\Sigma_s^t} \left[ \frac{1}{2}(\phi^{(j)})^2 \mathcal{B}_\phi(t)f + r_{j-1}(t)f\phi^{(j)} \right] d\Sigma. \end{aligned} \quad (3.24)$$

Next, we assume that the vector field  $H$  on  $\Omega$  satisfy the assumptions **H1** and **H2**. Noting that  $H$  does not depend on time  $t$ , we obtain, letting  $\mathcal{H} = H$  in the identity (3.23),

$$\Pi_{\wp_s^t} = \Pi_{\Sigma_s^t}, \quad (3.25)$$

where

$$\Pi_{\wp_s^t} = \int_{\wp_s^t} \left[ H(\phi^{(j)}) \phi_{v_B}^{(j)} + \frac{1}{2} (\dot{\phi}^{(j)2} - |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2) \langle H, v \rangle \right] d\wp, \quad (3.26)$$

$$\begin{aligned} \Pi_{\Sigma_s^t} &= (\dot{\phi}^{(j)}, H(\phi^{(j)}))_{L^2(\Omega)} \Big|_s^t - \int_{\Sigma_s^t} r_{j-1}(t) H(\phi^{(j)}) d\Sigma \\ &+ \int_{\Sigma_s^t} \left\{ D_{g_\phi} H(\nabla_{g_\phi} \phi^{(j)}, \nabla_{g_\phi} \phi^{(j)}) + \frac{1}{2} [(\dot{\phi}^{(j)})^2 - |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2] \operatorname{div} H \right\} d\Sigma. \end{aligned} \quad (3.27)$$

For obtaining the inequality (3.1), we shall estimate  $\Pi_{\wp_s^t}$  and  $\Pi_{\Sigma_s^t}$ , respectively.

*Estimate on  $\Pi_{\wp_s^t}$ .* Decompose  $\Pi_{\wp_s^t}$  as

$$\Pi_{\wp_s^t} = \Pi_{\wp_{1s}^t} + \Pi_{\wp_{0s}^t}$$

where  $\wp_{1s}^t = (s, t) \times \Gamma_1$  and  $\wp_{0s}^t = (s, t) \times \Gamma_0$ . On  $\wp_s^t = (s, t) \times \Gamma$  we have a decomposition of the direct sum in the metric  $g_\phi$

$$H = H_{\Gamma_{g_\phi}} + \left\langle H, \frac{B_\phi(t)v}{|B_\phi(t)v|_{g_\phi}} \right\rangle_{g_\phi} \frac{B_\phi(t)v}{|B_\phi(t)v|_{g_\phi}}. \quad (3.28)$$

The boundary condition  $\phi^{(j)} = 0$  on  $\Gamma_1$  implies that  $H_{\Gamma_{g_\phi}}(\phi^{(j)}) = 0$  for  $x \in \Gamma_1$ . By the relations (3.3) and the decomposition (3.28) yield

$$H(\phi^{(j)}) = \frac{\phi_{v_B}^{(j)}}{\langle B_\phi(t)v, v \rangle} \langle H, v \rangle, \quad \nabla_{g_\phi} \phi^{(j)} = \phi_{v_B}^{(j)} \frac{B_\phi(t)v}{\langle B_\phi(t)v, v \rangle}, \quad x \in \Gamma_1, \quad (3.29)$$

and therefore, we obtain

$$\Pi_{\wp_{1s}^t} = \frac{1}{2} \int_{\wp_{1s}^t} \frac{(\phi_{v_B}^{(j)})^2}{\langle B_\phi(t)v, v \rangle} \langle H, v \rangle d\wp \leq 0, \quad (3.30)$$

via the assumption (1.14) of Section 1. On  $\wp_{0s}^t = (s, t) \times \Gamma_0$ , the assumption (1.15) and the boundary formulas in (2.58) where  $l = j + 2$  yield, via the estimates in (2.28) and in Lemma 2.2, respectively,

$$\begin{aligned} \Pi_{\wp_{0s}^t} &\leq \int_{\wp_{0s}^t} \left[ H(\phi^{(j)}) \phi_{v_B}^{(j)} + \frac{1}{2} \dot{\phi}^{(j)2} \langle H, v \rangle \right] d\wp \\ &\leq \varepsilon \|\phi^{(j)}\|_{L^2((s,t), H^1(\Gamma_0))}^2 + c_{\gamma, \varepsilon} (\|\dot{\phi}^{(j)}\|_{L^2(\wp_{0s}^t)}^2 + \|\phi_{v_B}^{(j)}\|_{L^2(\wp_{0s}^t)}^2) \\ &\leq c_{\gamma, \varepsilon} \int_s^t [\mathcal{Q}_{\Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau)] d\tau + \varepsilon \|\phi^{(j)}\|_{L^2((s,t), H^1(\Gamma_0))}^2 + c_{\gamma, \varepsilon} \int_s^t \mathcal{L}(\tau) d\tau, \end{aligned} \quad (3.31)$$

where  $\varepsilon > 0$  can be small. In the above estimate, the following estimate is used: By the decomposition (3.28),

$$\begin{aligned} |H(\phi^{(j)})\phi_{v_B}^{(j)}| &\leq |H_{\Gamma_{g_\phi}}(\phi^{(j)})||\phi_{v_B}^{(j)}| + c_\gamma |\phi_{v_B}^{(j)}|^2 \\ &\leq c_\gamma |\nabla_\Gamma \phi^{(j)}||\phi_{v_B}^{(j)}| + c_\gamma |\phi_{v_B}^{(j)}|^2 \\ &\leq \varepsilon |\nabla_\Gamma \phi^{(j)}|^2 + c_{\varepsilon, \gamma} |\phi_{v_B}^{(j)}|^2, \quad x \in \Gamma_0, \end{aligned}$$

where  $\nabla_\Gamma$  is the gradient of the induced metric on  $\Gamma$  from the dot metric of  $\mathbb{R}^n$ .

We now estimate the term  $\|\phi^{(j)}\|_{L^2((s,t), H^1(\Gamma_0))}^2$ . Let  $H_1$  be a vector field on  $\overline{\Omega^2}$  such that

$$H_1 = 0, \quad x \in \Gamma_1, \quad H_1 = B_\phi(t)v, \quad x \in \Gamma_0. \quad (3.32)$$

Then Since the vector field  $\frac{B_\phi(t)v}{\langle B_\phi(t)v, v \rangle^{1/2}}$  is the unit normal of the metric  $g_\phi$  along the boundary  $\Gamma$ , by the relations (3.3), we have a directional decomposition

$$\begin{aligned} \nabla_{g_\phi} \phi^{(j)} &= \nabla_{\Gamma_{g_\phi}} \phi^{(j)} + \left\langle \nabla_{g_\phi} \phi^{(j)}, \frac{B_\phi(t)v}{\langle B_\phi(t)v, v \rangle^{1/2}} \right\rangle_{g_\phi} \frac{B_\phi(t)v}{\langle B_\phi(t)v, v \rangle^{1/2}} \\ &= \nabla_{\Gamma_{g_\phi}} \phi^{(j)} + \phi_{v_B}^{(j)} \frac{B_\phi(t)v}{\langle B_\phi(t)v, v \rangle}, \end{aligned} \quad (3.33)$$

where  $\nabla_{\Gamma_{g_\phi}}$  is the gradient of the induced metric of the boundary  $\Gamma$  from the metric  $g_\phi$ . It follows from (3.33) that

$$|\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2 = |\nabla_{\Gamma_{g_\phi}} \phi^{(j)}|_{g_\phi}^2 + (\phi_{v_B}^{(j)})^2 / \langle B_\phi(t)v, v \rangle. \quad (3.34)$$

We replace the vector field  $\mathcal{H}$  in the identity (3.23) with  $-H_1$ , given by (3.32), and, noting that  $H_1 = 0$  on  $\Gamma_1$  and  $H_1(\phi^{(j)}) = \phi_{v_B}^{(j)}$  on  $\Gamma_0$ , we obtain, by (3.34),

$$\text{the left-hand side of (3.23)} = \frac{1}{2} \int_{\mathcal{S}^{n-1}_s} [|\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2 - (\dot{\phi}^{(j)})^2] \langle B_\phi(t)v, v \rangle d\mathcal{G} \quad (3.35)$$

and

$$\begin{aligned} \text{the right-hand side of (3.23)} &\leq c_\gamma \vartheta_j(s) + c_\gamma \vartheta_j(t) + c_\gamma \int_s^t \left[ \vartheta_j(\tau) + \sum_{k=2}^{m-1} \mathcal{E}^{k/2}(\tau) \vartheta_j^{1/2}(\tau) \right] d\tau \\ &\leq c_\gamma Q(s) + c_\gamma Q(t) + c_\gamma \int_s^t (Q(\tau) + \mathcal{L}(\tau)) d\tau \end{aligned} \quad (3.36)$$

for  $1 \leq j \leq m-1$ , where  $\vartheta_j(t)$  is given by (2.61) and  $Q(t) = 2\lambda \sum_{j=0}^{m-1} \vartheta_j(t)$ . In addition, the equations in (3.22) on the boundary  $(0, T) \times \Gamma_0$  yield

$$\int_s^t \|\dot{\phi}^{(j)}\|_{L^2(\Gamma_0)}^2 d\tau, \int_s^t \|\phi_{v_B}^{(j)}\|_{L^2(\Gamma_0)}^2 d\tau \leq c \int_s^t (Q_{I, \Gamma_0}(\tau) + Q_{O, \Gamma_0}(\tau)) d\tau, \quad (3.37)$$

where  $Q_{I, \Gamma_0}(t)$  and  $Q_{O, \Gamma_0}(t)$  are given by (2.19) and (2.20), respectively. Combining the relations (3.35)–(3.37) and the identity (3.23) give the estimate

$$\begin{aligned} \|\phi^{(j)}\|_{L^2((s,t), H^1(\Gamma_0))}^2 &\leq c_\gamma Q(s) + c_\gamma Q(t) \\ &\quad + c_\gamma \int_s^t (Q(\tau) + Q_{I, \Gamma_0}(\tau) + Q_{O, \Gamma_0}(\tau) + \mathcal{L}(\tau)) d\tau. \end{aligned} \quad (3.38)$$

Moreover, the inequalities (2.25) and (2.26) of Section 2 imply that

$$\max\{Q(t), Q(s)\} \leq \frac{1}{t-s} \int_s^t Q(\tau) d\tau + c_\gamma \int_s^t [Q_{I, \Gamma_0}(\tau) + Q_{O, \Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau. \quad (3.39)$$

Inserting (3.39) into (3.38), and then inserting (3.38) into (3.31) yield

$$\Pi_{\wp_s^t} \leq \varepsilon c_\gamma [1 + 1/(t-s)] \int_s^t Q(\tau) d\tau + c_{\gamma, \varepsilon} \int_s^t [Q_{I, \Gamma_0}(\tau) + Q_{O, \Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau, \quad (3.40)$$

where  $\varepsilon > 0$  can be small and  $0 \leq s \leq t \leq T$ .

*Estimate on  $\Pi_{\Sigma_s^t}$ .* It follows from the identity (3.10) and the inequality (3.9) in Lemma 3.1 and the assumption in (1.13) that

$$\begin{aligned} D_{g_\phi} H(\nabla_{g_\phi} \phi^{(j)}, \nabla_{g_\phi} \phi^{(j)}) &= D_g H(\nabla_{g_\phi} \phi^{(j)}, \nabla_{g_\phi} \phi^{(j)}) + \eta(\nabla_{g_\phi} \phi^{(j)}, \nabla_{g_\phi} \phi^{(j)}) \\ &\geq \rho_0 |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2 - c_\gamma \mathcal{L}(t) \\ &\geq \rho_0 c_{0, \gamma} |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2 - c_\gamma \mathcal{L}(t). \end{aligned} \quad (3.41)$$

Noting that

$$\int_\Omega |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2 dx = (B_\phi(t) \nabla \phi^{(j)}, \nabla \phi^{(j)})_{L^2(\Omega)},$$

we have, by (3.27) and (3.41),

$$\begin{aligned}
\Pi_{\Sigma_s^t} &\geq \rho_0 c_{0,\gamma} \int_{\Sigma_s^t} |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2 d\Sigma + \frac{1}{2} \int_{\Sigma_s^t} [(\dot{\phi}^{(j)})^2 - |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2] \operatorname{div} H d\Sigma \\
&\quad - c_\gamma \mathcal{Q}(s) - c_\gamma \mathcal{Q}(t) - c_\gamma \int_s^t \mathcal{L}(\tau) d\tau \\
&\geq \frac{\rho_0 c_{0,\gamma}}{2} \int_s^t \vartheta_j(\tau) d\tau + \int_{\Sigma_s^t} f(\dot{\phi}^{(j)2} - |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2) d\Sigma \\
&\quad - c_\gamma \mathcal{Q}(s) - c_\gamma \mathcal{Q}(t) - c_\gamma \int_s^t \mathcal{L}(\tau) d\tau,
\end{aligned} \tag{3.42}$$

where  $\vartheta_j(t)$  is given in (2.61) of Section 2 and

$$f = \frac{\operatorname{div} H - \rho_0 c_{0,\gamma}}{2}. \tag{3.43}$$

On the other hand, using the identity (3.24), where  $f$  is given in (3.43), and the inequality (2.28), we obtain

$$\begin{aligned}
&\left| \int_{\Sigma_s^t} f(\dot{\phi}^{(j)2} - |\nabla_{g_\phi} \phi^{(j)}|_{g_\phi}^2) d\Sigma \right| \\
&\leq c_\gamma [\mathcal{Q}(s) + \mathcal{Q}(t)] + c_\gamma (\|\phi^{(j)}(s)\|_{L^2(\Omega)}^2 + \|\phi^{(j)}(t)\|_{L^2(\Omega)}^2) + c_\gamma \int_s^t \|\phi^{(j)}(\tau)\|_{L^2(\Omega)}^2 d\tau \\
&\quad + c_\gamma \int_s^t (\|\phi^{(j)}\|_{L^2(\Gamma_0)}^2 + \|\phi_{\nu_B}^{(j)}\|_{L^2(\Gamma_0)}^2) d\tau + c_\gamma \int_s^t \mathcal{L}(\tau) d\tau.
\end{aligned} \tag{3.44}$$

Inserting (3.44) into (3.42), and using the relations (3.25), (3.39), (3.40), give

$$\begin{aligned}
&\int_s^t \vartheta_j(\tau) d\tau \leq \varepsilon c_\gamma [1 + 1/(t-s)] \int_s^t \mathcal{Q}(\tau) d\tau + c_{\varepsilon,\gamma} \int_s^t [\mathcal{Q}_{\Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O},\Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau \\
&\quad + c_{\varepsilon,\gamma} \Upsilon(s) + c_{\varepsilon,\gamma} \Upsilon(t) + c_{\varepsilon,\gamma} \int_s^t \Upsilon(\tau) d\tau,
\end{aligned} \tag{3.45}$$

for  $1 \leq j \leq m-1$ , where

$$\Upsilon(t) = \sum_{j=0}^{m-1} \|\phi^{(j)}(t)\|_{L^2(\Omega)}^2 \tag{3.46}$$

is a lower order term with respect to  $\mathcal{E}(t)$  and the term  $\int_s^t \|\phi_{v_B}^{(j)}(\tau)\|_{L^2(\Gamma_0)}^2 d\tau$  is bounded in (3.37).

Furthermore, by the definition (3.46), we have

$$|\dot{\Upsilon}(t)| = 2 \left| \sum_{k=0}^{m-1} (\phi^{(k+1)}, \phi^{(k)})_{L^2(\Omega)} \right| \leq \varepsilon \mathcal{Q}(t) + c_\varepsilon \Upsilon(t),$$

which implies

$$\max\{\Upsilon(s), \Upsilon(t)\} \leq \varepsilon \int_s^t \mathcal{Q}(\tau) d\tau + [c_\varepsilon + 1/(t-s)] \int_s^t \Upsilon(\tau) d\tau \quad (3.47)$$

for  $0 \leq s < t \leq T$ .

Inserting (3.47) into (3.45), we obtain constants  $c_\gamma > 0$ ,  $c_{\varepsilon, \gamma} > 0$  and  $T_\gamma > 3 \sup_{x \in \bar{\Omega}} |H|_g / \rho_0$  such that, if  $0 \leq s < t \leq T$  and  $t - s \geq T_\gamma$ , then

$$\int_s^t \vartheta_j(\tau) d\tau \leq \varepsilon c_\gamma \int_s^t \mathcal{Q}(\tau) d\tau + c_{\varepsilon, \gamma} \int_s^t [\mathcal{Q}_{1, \Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) + \mathcal{L}(\tau) + \Upsilon(\tau)] d\tau, \quad (3.48)$$

for  $1 \leq j \leq m-1$  and  $\varepsilon > 0$  small.

**Step II.** This time we consider a metric

$$h_\phi = N_\phi^{-1}(t), \quad (3.49)$$

where the matrix  $N_\phi(t)$  is given by (2.13), to replace the metric  $g_\phi$  in (3.2), and use the system (2.14)–(2.15) instead of the system (3.22). By repeating the procedure of Step I, we obtain that the inequality (3.48) holds when  $\vartheta_j(t)$  is replaced with  $\vartheta_0(t)$ , where  $\vartheta_0(t)$  is given by (2.60).

Now, we sum up both the sides of the inequality (3.48) from  $j = 0$  to  $j = m-1$ , move the first term of the right-hand side to the left-hand side and obtain a constant  $c_\gamma > 0$  such that

$$\int_s^t \mathcal{Q}(\tau) d\tau \leq c_\gamma \int_s^t [\mathcal{Q}_{1, \Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) + \mathcal{L}(\tau) + \Upsilon(\tau)] d\tau. \quad (3.50)$$

Finally the inequality (3.1) follows from (3.50) and Lemma 3.2 below.  $\square$

**Lemma 3.2.** *Let all the assumptions in Theorem 1.1 hold. Suppose that  $c_\gamma > 0$  and  $T_\gamma > 3 \sup_{x \in \bar{\Omega}} |H|_g / \rho_0$  such that the inequality (3.48) holds. If  $0 \leq s < t \leq T$  and  $t - s \geq T_\gamma$ , then*

$$\int_s^t \Upsilon(\tau) d\tau \leq c_\gamma \int_s^t [\mathcal{Q}_{1, \Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau \quad (3.51)$$

where  $\Upsilon(t)$  is given by (3.46).



**Proof.** By Lemma 4 in [7] (or the proof of Theorem 4.1 in [29]), we only need to prove that there is  $c_\gamma > 0$  such that

$$\int_0^{T_\gamma} \Upsilon(\tau) d\tau \leq c_\gamma \int_0^{T_\gamma} [\mathcal{Q}_{\mathbf{I}, \Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) + \mathcal{L}(\tau)] d\tau. \quad (3.52)$$

By contradiction. Suppose that (3.52) does not hold for some  $\gamma_0 > 0$ . Then, there exist initial data  $(\phi_0^k, \phi_1^k)$ , inputs  $\mathbf{I}_k$ , and the corresponding solutions  $\phi^k$  of the problem (2.1) over  $[0, T]$  such that

$$\sup_{0 \leq t \leq T} \|\phi^k\|_{H^m(\Omega)} \leq \gamma_0, \quad (3.53)$$

$$\int_0^{T_{\gamma_0}} \Upsilon_k(\tau) d\tau \geq k \int_0^{T_{\gamma_0}} [\mathcal{Q}_{\mathbf{I}_k, \Gamma_0}(\tau) + \mathcal{Q}_{\mathcal{O}_k, \Gamma_0}(\tau) + \mathcal{L}_k(\tau)] d\tau \quad (3.54)$$

where

$$\Upsilon_k(t) = \sum_{j=0}^{m-1} \|(\phi^k)^{(j)}(t)\|_{L^2(\Omega)}^2, \quad \mathcal{L}_k(t) = \sum_{j=3}^{2m} \mathcal{E}_k^{j/2}(t), \quad \mathcal{E}_k(t) = \sum_{j=0}^m \|(\phi^k)^{(j)}\|_{H^{m-j}(\Omega)}^2$$

for  $k \geq 1$ . Since

$$\int_0^{T_{\gamma_0}} \Upsilon_k(\tau) d\tau \leq \int_0^{T_{\gamma_0}} \mathcal{E}_k(\tau) d\tau \leq T_{\gamma_0}^{1/2} \left( \int_0^{T_{\gamma_0}} \mathcal{L}_k(\tau) d\tau \right)^{1/2},$$

we obtain, by (3.54),  $\int_0^{T_{\gamma_0}} \mathcal{L}_k(\tau) d\tau \rightarrow 0$ , which, in turn, implies that  $\int_0^{T_{\gamma_0}} \mathcal{E}_k(\tau) d\tau \rightarrow 0$  as  $k \rightarrow \infty$ , i.e.,

$$\phi^k \rightarrow 0 \quad \text{in } H^m((0, T_{\gamma_0}) \times \Omega).$$

It follows by the embedding theorem

$$\sup_{(t,x) \in [0, T_{\gamma_0}] \times \bar{\Omega}} |\phi^k| \rightarrow 0 \quad (3.55)$$

as  $k$  goes to infinity.

Set

$$c_k^2 = \int_0^{T_{\gamma_0}} \Upsilon_k(\tau) d\tau, \quad \psi_k = \phi^k / c_k, \quad \theta_k = \mathbf{I}_k / c_k, \quad \eta_k = \mathcal{O}_k / c_k. \quad (3.56)$$

Then

$$\sum_{j=0}^{m-1} \int_0^{T_{\gamma_0}} \|\psi_k^{(j)}(\tau)\|_{L^2(\Omega)}^2 d\tau = 1 \quad (3.57)$$

and by (3.54)

$$\begin{aligned} & \int_0^{T_{\gamma_0}} (\|\theta_k - \lambda \langle \mathbf{a}(x, \nabla w), v \rangle / c_k\|_{L^2(\Gamma_0)}^2 + \|\eta_k + \lambda \langle \mathbf{a}(x, \nabla w), v \rangle / c_k\|_{L^2(\Gamma_0)}^2) d\tau \\ & + \sum_{j=1}^{m-1} \int_0^{T_{\gamma_0}} (\|\theta_k^{(j)}\|_{L^2(\Gamma_0)}^2 + \|\eta_k^{(j)}\|_{L^2(\Gamma_0)}^2) d\tau + \frac{1}{c_k^2} \sum_{j=3}^{2m} \int_0^{T_{\gamma_0}} \mathcal{E}_k^{j/2}(\tau) d\tau \rightarrow 0 \end{aligned} \quad (3.58)$$

as  $k \rightarrow \infty$ .

We divide both the sides of the inequality (3.50) with  $s = 0$  and  $t = T_{\gamma_0}$  by  $c_k^2$  to see that

$$\begin{aligned} & \int_0^{T_{\gamma_0}} [\|\dot{\psi}_k(\tau)\|_{L^2(\Omega)}^2 + (N_\phi(\tau) \nabla \psi_k, \nabla \psi_k)_{L^2(\Omega)}] d\tau \\ & + \sum_{j=1}^{m-1} \int_0^{T_{\gamma_0}} [\|\psi_k^{(j+1)}(\tau)\|_{L^2(\Omega)}^2 + (B_\phi(\tau) \nabla \psi_k^{(j)}(\tau), \nabla \psi_k^{(j)}(\tau))_{L^2(\Omega)}] d\tau \end{aligned} \quad (3.59)$$

are bounded for all  $k \geq 1$ . Then there is  $\varphi \in H^1((0, T_{\gamma_0}) \times \Omega)$  such that

$$\psi_k^{(j)} \rightharpoonup \varphi^{(j)} \quad \text{weakly in } H^1((0, T_{\gamma_0}) \times \Omega), \quad (3.60)$$

$$\psi_k^{(j)} \rightarrow \varphi^{(j)} \quad \text{strongly in } L^2((0, T_{\gamma_0}) \times \Omega), \quad (3.61)$$

for  $0 \leq j \leq m-1$  as  $k$  goes to infinity.

Next, we divide the formulas for  $1 \leq j \leq m-1$  in (3.22) and in (2.14) of Section 2 for  $j = 0$ , respectively, by  $c_k$  and use the estimates (2.34) and (3.58) to show that  $\varphi \in H^1((0, T_{\gamma_0}) \times \Gamma)$  satisfies for  $(t, x) \in (0, T_{\gamma_0}) \times \Omega$

$$\begin{cases} \ddot{\varphi} = \mathcal{A}\varphi, & (t, x) \in (0, T_{\gamma_0}) \times \Omega, \\ \varphi = 0, & (t, x) \in (0, T_{\gamma_0}) \times \Gamma_1, \\ \dot{\varphi} + \lambda \varphi_{v_A} = 0, & (t, x) \in (0, T_{\gamma_0}) \times \Gamma_0, \\ \dot{\varphi} - \lambda \varphi_{v_A} = 0, & (t, x) \in (0, T_{\gamma_0}) \times \Gamma_0, \end{cases} \quad (3.62)$$

where

$$\mathcal{A}\varphi = \operatorname{div} A(x, \nabla w) \nabla \varphi, \quad \varphi_{v_A} = \langle A(x, \nabla w) \nabla \varphi, v \rangle. \quad (3.63)$$

By observability inequalities for the linear wave equation, for example, [25, Theorem 1.1], the formulas (3.62) imply that

$$\varphi^{(j)} = 0, \quad 1 \leq j \leq m-1. \quad (3.64)$$

Then  $\varphi$  is a constant. If  $\Gamma_1 \neq \emptyset$ , then  $\varphi|_{\Gamma_1} = 0$  implies that  $\varphi = 0$  on  $\overline{\Omega}$ . Let  $\Gamma_1 = \emptyset$ . By the assumption (1.16) and (3.55),

$$\int_{\Omega} w \, dx = \lim_{k \rightarrow \infty} \int_{\Omega} \phi_0^k \, dx = 0.$$

We obtain, by (1.16) again,

$$\int_{\Omega} \varphi \, dx = \lim_{k \rightarrow \infty} \int_{\Omega} \frac{\phi_0^k}{c_k} \, dx = 0,$$

that is  $\varphi = 0$  on  $\overline{\Omega}$ . These contradict (3.57).  $\square$

By a similar argument as in Lemma 3.2, the inequality of the right-hand side in (2.24) of Theorem 2.1 in Section 2 can be improved into

**Lemma 3.3.** *Let all the assumptions in Theorem 1.1 hold. Let  $\gamma > 0$  be given and  $\phi$  satisfy the problem (2.1) on the interval  $[0, T]$  for some  $T > 3 \sup_{x \in \overline{\Omega}} |H|_g / \rho_0$  such that (2.23) is true. Then there is  $c_\gamma > 0$  such that*

$$\mathcal{E}(t) \leq c_\gamma \mathcal{Q}(t) + c_\gamma \mathcal{E}_{I, \Gamma_0}(t) + c_\gamma \mathcal{L}(t) \quad (3.65)$$

for  $T \geq t \geq 3 \sup_{x \in \overline{\Omega}} |H|_g / \rho_0$ .

#### 4. Global smooth solutions

**Proof of Theorem 1.1.** Let  $(\phi_0, \phi_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$  and  $I \in \mathcal{I}^m((0, \infty), \Gamma_0)$  be given such that the problem (2.1) has a short time solution. We shall look for a constant  $\eta_0 > 0$  such that, if

$$\mathcal{E}(0) + \|\hat{I}\|_{\mathcal{I}^m((0, \infty), \Gamma_0)}^2 \leq \eta_0, \quad (4.1)$$

where

$$\hat{I} = I - \langle \mathbf{a}(x, \nabla w), v \rangle, \quad (4.2)$$

then the solution of the problem (2.1) is global.

Through this section we take

$$\gamma = 1.$$

Let

$$c_{0,1} > 0, \quad c_1 \geq 1, \quad T_1 > 3 \sup_{x \in \overline{\Omega}} |H|_g / \rho_0$$

be given such that all the corresponding estimates in Sections 2, 3 hold, respectively, according to  $\gamma = 1$ . Let

$$\mathcal{E}(\eta) = \sum_{k=0}^{2m-3} \eta^{k/2}, \quad \eta \in (0, \infty), \quad (4.3)$$

$$\Theta(t) = 2e^\eta \max\{c_1 c_{0,1}^{-1} + 2c_1 \lambda, 5c_1^2, 3c_1^2 \mathcal{E}(1)\eta\}, \quad (4.4)$$

for  $\eta \in (0, \infty)$ .

Let

$$0 < \eta < \min\{1, \Theta^{-2}(2T_1)/4\} \quad (4.5)$$

be given. We assume that

$$\mathcal{E}(0) + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((0,\infty),\Gamma_0)}^2 \leq \eta^{3/2} < \eta. \quad (4.6)$$

Then there is some  $\delta > 0$  such that

$$\mathcal{E}(t) < \eta \quad (4.7)$$

for  $t \in [0, \delta)$ . Let  $\delta_0 > 0$  be the largest number such that the inequality (4.7) holds true for  $t \in [0, \delta_0)$ .

**Step I.** We have

$$\delta_0 > 2T_1. \quad (4.8)$$

Indeed, letting (4.8) be not true, i.e.,  $\delta_0 \leq 2T_1$ , we shall have a contradiction as follows. Using (2.18), (2.17) and (2.24), we have

$$P(0) \leq 2(c_{0,1}^{-1} + 2\lambda)\mathcal{E}(0), \quad (4.9)$$

which gives, by (2.27),

$$\begin{aligned} \|\phi(t)\|_{L^2(\Omega)}^2 &\leq 2\{(c_{0,1}^{-1} + 2\lambda)\mathcal{E}(0) + c_1 \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((0,\infty),\Gamma_0)}^2 + 2c_1 T_1 \eta^{3/2} \mathcal{E}(1)\} e^{2T_1} \\ &\leq 2e^{2T_1} \max\{c_{0,1}^{-1} + 2\lambda, c_1, c_1 T_1 \mathcal{E}(1)\} [\mathcal{E}(0) + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((0,\infty),\Gamma_0)}^2 + \eta^{3/2}] \end{aligned} \quad (4.10)$$

since

$$L(t) \leq \eta^{3/2} \mathcal{E}(\eta) \leq \eta^{3/2} \mathcal{E}(1), \quad 0 \leq t \leq \delta_0$$

by (4.7). In addition, we multiply (2.25) by 2, add it to (2.26), then integrate it over  $(0, 2T_1)$ , and obtain

$$\mathcal{Q}(t) \leq \max\{c_{0,1}^{-1}, 10, 6c_1 T_1 \mathcal{E}(1)\} [\mathcal{E}(0) + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((0,\infty), \Gamma_0)}^2 + \eta^{3/2}]. \quad (4.11)$$

Inserting (4.11) and (4.10) into the right-hand side of the inequality (2.24) yields, via (4.6) and (4.5),

$$\begin{aligned} \mathcal{E}(t) &\leq \Theta(2T_1) [\mathcal{E}(0) + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((0,t), \Gamma_0)}^2 + \eta^{3/2}] \\ &\leq 2\Theta(2T_1) \eta^{1/2} \eta < \eta, \quad 0 \leq t \leq \delta_0. \end{aligned} \quad (4.12)$$

This contradicts the definition of  $\delta_0$ .

**Step II.** Let  $T_1 \leq s < t < \delta_0$  with  $t - s \geq T_1$ . Integrating the inequality (3.65) over  $(s, t)$  yields

$$\int_s^t \mathcal{E}(\tau) d\tau \leq c_1 \int_s^t \mathcal{Q}(\tau) d\tau + c_1 \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 + c_1 \mathcal{E}(1) \eta^{1/2} \int_s^t \mathcal{E}(\tau) d\tau,$$

that is,

$$\int_s^t \mathcal{E}(\tau) d\tau \leq c_1 [1 - c_1 \mathcal{E}(1) \eta^{1/2}]^{-1} \left( \int_s^t \mathcal{Q}(\tau) d\tau + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 \right), \quad (4.13)$$

where

$$\|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 = \max_{s \leq \tau < t} \mathcal{E}_{1, \Gamma_0}(\tau) + \int_s^t \mathcal{E}_{1, \Gamma_0}(\tau) d\tau + \int_s^t \|\mathbf{I}^{(m-1)}(\tau)\|_{L^2(\Gamma_0)}^2 d\tau.$$

The inequality (4.13), in turn, gives

$$\int_s^t \mathcal{L}(\tau) d\tau \leq f(\eta^{1/2}) \left( \int_s^t \mathcal{Q}(\tau) d\tau + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 \right), \quad (4.14)$$

where

$$f(x) = c_1 \mathcal{E}(1) x [1 - c_1 \mathcal{E}(1) x]^{-1}, \quad 0 < x < 1. \quad (4.15)$$

Furthermore, by combining the inequalities (3.1) and (4.14), we obtain

$$\begin{aligned} \int_s^t \mathcal{Q}(\tau) d\tau &\leq c_1 [1 - c_1 f(\eta^{1/2})]^{-1} [1 + f(\eta^{1/2})] \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 \\ &\quad + c_1 [1 - c_1 f(\eta^{1/2})]^{-1} \int_s^t \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) d\tau. \end{aligned} \quad (4.16)$$

On the other hand, we integrate the inequality (2.25) over  $(s, t)$  and obtain, via (4.14),

$$\begin{aligned} \mathcal{Q}(t) &\leq \mathcal{Q}(s) + [4 + c_1 f(\eta^{1/2})] \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 \\ &\quad + c_1 f(\eta^{1/2}) \int_s^t \mathcal{Q}(\tau) d\tau - \int_s^t \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) d\tau. \end{aligned} \quad (4.17)$$

Let  $\kappa > 0$  be given. We multiply (4.16) by  $\kappa$ , then add to (4.17), and have

$$\begin{aligned} \mathcal{Q}(t) + [\kappa - c_1 f(\eta^{1/2})] \int_s^t \mathcal{Q}(\tau) d\tau \\ \leq \mathcal{Q}(s) + \{\kappa c_1 [1 - c_1 f(\eta^{1/2})]^{-1} [1 + f(\eta^{1/2})] + 4 + c_1 f(\eta^{1/2})\} \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 \\ + \{\kappa c_1 [1 - c_1 f(\eta^{1/2})]^{-1} - 1\} \int_s^t \mathcal{Q}_{\mathcal{O}, \Gamma_0}(\tau) d\tau \end{aligned} \quad (4.18)$$

for  $T_1 \leq s \leq t \leq \delta_0$  with  $t - s \geq T_1$ . Letting  $\kappa = [1 - c_1 f(\eta^{1/2})]/c_1$  in (4.18) gives

$$\mathcal{Q}(t) + \omega(\eta) \int_s^t \mathcal{Q}(\tau) d\tau \leq \mathcal{Q}(s) + [5 + (1 + c_1) f(\eta^{1/2})] \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((s,t), \Gamma_0)}^2 \quad (4.19)$$

for  $T_1 \leq s \leq t \leq \delta_0$  with  $t - s \geq T_1$ , where

$$\omega(\eta) = c_1^{-1} - (1 + c_1) f(\eta^{1/2}). \quad (4.20)$$

Let

$$\tau_0 = \max\{c_1 c_{0,1}^{-1} \Theta(2T_1), c_1(7 + c_1), c_1 \mathcal{E}(1)\}.$$

We fix  $\eta > 0$  small such that

$$\eta^{1/2} < \frac{1}{2\tau_0}, \quad \omega(\eta) > 0. \quad (4.21)$$

Then when the condition (4.6) holds, the inequalities (3.65), (4.12), and (4.19) imply that the estimate (4.7) holds with  $\delta_0 = \infty$ . Indeed, let  $\delta_0 < \infty$  and we shall have a contradiction as follows. The condition  $2c_1 \mathcal{E}(1)\eta < 1$  implies  $f(\eta^{1/2}) \leq 1$ . Then using (4.19) with  $s = T_1$  in (3.65), we have

$$\mathcal{E}(t) \leq c_1 c_{0,1}^{-1} \mathcal{E}(T_1) + c_1(7 + c_1) \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((T_1,t), \Gamma_0)}^2 + c_1 \mathcal{E}(1) \eta^{3/2} \quad (4.22)$$

for  $0 \leq t \leq \delta_0$ . Finally, we use the inequality (4.12) with  $t = T_1$  in (4.22) and obtain, by (4.6) and (4.21),

$$\begin{aligned}\mathcal{E}(t) &\leq \tau_0 [\mathcal{E}(0) + \|\hat{\mathbf{I}}\|_{\mathcal{I}^m((0,t),\Gamma_0)}^2 + \eta^{3/2}] \\ &\leq 2\tau_0 \eta^{1/2} \eta < \eta\end{aligned}$$

for all  $T_1 \leq t \leq \delta_0$ , which contradicts the definition of  $\delta_0$  again.  $\square$

**Proof of Theorem 1.2.** Let  $T_0 > 0$  be such that

$$\mathbf{I}(t) = \lambda \langle \mathbf{a}(x, \nabla w), v \rangle, \quad x \in \Gamma_0, \quad t \geq T_0. \quad (4.23)$$

By (4.19), we have

$$\mathcal{Q}(t) + \omega(\eta) \int_s^t \mathcal{Q}(\tau) d\tau \leq \mathcal{Q}(s) \quad (4.24)$$

for  $\max\{T_0, T_1\} \leq s \leq t < \infty$  with  $t - s \geq T_1$ . Let  $\omega_0 = \max\{T_0, T_1\}$  and  $t > \omega_0$ . We multiply (4.24) by  $s^k$ , integrate it from 0 to  $t - \omega_0$  and obtain

$$\int_0^{t-\omega_0} s^k \mathcal{Q}(s) ds \geq \frac{(t-\omega_0)^{k+1}}{k+1} \mathcal{Q}(t) + \frac{\omega(\eta)}{k+1} \int_0^{t-\omega_0} \tau^{k+1} \mathcal{Q}(\tau) d\tau \quad (4.25)$$

for all  $k \geq 0$  which yield

$$\mathcal{Q}(t) \leq \mathcal{Q}(0) e^{-\omega(\eta)(t-\omega_0)}, \quad t \geq \omega_0. \quad (4.26)$$

The estimate (1.17) follows from (3.65) and (4.26) since the condition (4.23) implies that  $\mathcal{E}_{1,\Gamma_0}(t) = 0$  for  $t \geq T_0$ .  $\square$

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